Channel Design to Minimize Lining Material Costs

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Abstract: A direct algebraic technique is developed to determine open channel cross-sectional designs which minimize lining material costs when base and side wall unit costs are different. Solution graphs, which indicate both the optimal parameter combination and the costs of deviating from the optimal design, are presented. The graphs or analytical technique are also effective in designing any trapezoidal channels. Use of the technique shows that moderate deviations from optimal designs are not costly.

Introduction

The material cost of channel linings depends upon the volume of lining material used. On a per unit length basis, the material costs are a function of the lining thickness and perimeter length, which, in turn, depends upon the channel cross-sectional shape. The capacity of a channel constructed on a given slope will also be a function of the cross-sectional shape. Using this interrelationship, cross sections that minimize lining material costs can be determined.

If the channel is lined with one material of uniform thickness, the problem reduces to one of determining the best hydraulic section (1). However, if material or lining thickness is changed along the perimeter, the problem becomes much more complex. This paper will present a solution to the cost minimization problem when the material cost per unit surface area of the base of trapezoidal (or rectangular) channels is different from the material cost of the sides.

This analysis does not deal with placement or other construction costs. If they can be evaluated in terms of surface area, they can be combined with material costs. Otherwise, they must be considered separately. The technique will thus be most useful when long channel sections are to be constructed, allowing construction procedures to be oriented toward minimizing material costs or when labor costs are low relative to material costs, such as is the case in developing countries.

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Basic Equations

Hydraulic Relationships.—The capacity of a uniformly flowing open channel can be determined by Manning's equation

$$Q = \frac{c}{n} AR^{2/3} S^{1/2} \quad \text{(1)}$$

in which Q = flow rate; c = a constant equal to 1.486 when U.S. Customary units are used and 1.0 when SI units are used; n = Manning's roughness coefficient; A = cross-sectional flow area; R = hydraulic radius = A/P; P = wetted perimeter length; and S = slope. For a given channel, Q, n, and S are generally established by the given conditions (i.e., design flow rate, construction materials and techniques, and topography). The design parameters the engineer must determine are A and R, such that

$$AR^{2/3} = \frac{Qn}{cS^{1/2}} \quad \text{(2)}$$

The left side of Eq. 2 is called the section factor (1) and will be used to indicate the geometric capacity of the channel. The objective will be to minimize costs for a given required section factor.

The most commonly constructed lining cross section is the trapezoid. (Rectangular and triangular cross sections are specific cases of the trapezoidal section). The section factor for a trapezoid is

$$AR^{2/3} = \frac{A^{5/3}}{P^{2/3}} = \frac{(BD + zD^2)^{5/3}}{[B + 2D(z^2 + 1)^{1/2}]^{2/3}} \quad \text{(3)}$$

in which z = the side slope (horizontal to vertical); B = the bottom width; and D = the normal flow depth. Solving for the depth, D, and combining with Eq. 2 gives

$$D = \frac{\left[\frac{B}{D} + 2(z^2 + 1)^{1/2}\right]^{1/4}}{\left(\frac{B}{D} + z\right)^{5/8}} \left(\frac{Qn}{cS^{1/2}}\right)^{3/8} \quad \text{(4)}$$

If the B/D ratio and a side slope is specified, the geometric parameters can be determined directly. For instance, the best hydraulic section for a trapezoidal channel has a B/D ratio of (3)

$$\frac{B}{D} = 2(z^2 + 1)^{1/2} - 2z \quad \text{(5)}$$

Thus, the minimum material cost cross-section for lining with one material of uniform thickness would be when (substituting Eq. 5 into Eq. 4)

$$D = \frac{[4(z^2 + 1)^{1/2} - 2z]^{1/4}}{[2(z^2 + 1)^{1/2} - z]^{5/8}} \left(\frac{Qn}{cS^{1/2}}\right)^{3/8} \quad \text{(6)}$$

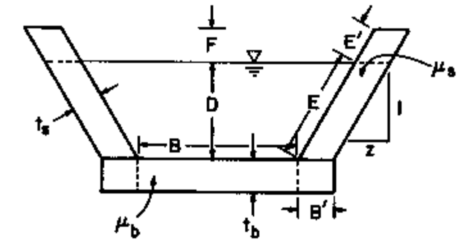

FIG. 1.—Definition of Dimensional Terms Used in Cost Calculations for Trapezoidal Channels

and B is as given by Eq. 5, for any selected side slope, specified roughness, and required channel slope and flow rate.

Cost Equations.—Material costs are usually a function of the volume of material used. If thickness requirements of the respective materials are specified, then costs on a unit channel length basis will be a function of length of the wetted perimeter plus freeboard and corners constructed of that material. Fig. 1 shows the terms used in the cost calculation for a trapezoidal channel (Eqs. 7, 8, and 9) where the base and side wall material costs are different.

$$C_b = \mu_b \times \frac{\text{volume}}{\text{unit length}} = \mu_b \times t_b \times (B + 2B') = bB + k \quad \text{(7)}$$

$$C_s = \mu_s \times \frac{\text{volume}}{\text{unit length}} = \mu_s \times t_s \times (2E + 2E') = 2d(D + F)(z^2 + 1)^{1/2} \quad \text{(8)}$$

$$C = C_b + C_s = bB + 2d(D + F)(z^2 + 1)^{1/2} + k \quad \text{(9)}$$

in which C = the total material cost per unit length; C_b = material cost of base per unit channel length; C_s = material cost of sides per unit channel length; B = bottom width; B' = bottom corner width; t_b = bottom lining thickness; D = flow depth; F = freeboard allowance; t_s = side lining thickness; E = wetted side length; E' = freeboard side length allowance; z = side slope (horizontal to vertical); μ_b = cost of base material per unit volume; b = cost of base material at a given thickness, t_b, per unit area; k = cost of corner materials per unit channel length; μ_s = cost of side material per unit volume; and d = cost of side material at a given thickness, t_s, per unit area.

Solution Formulation

In order to determine minimum cost trapezoidal cross sections when the per unit area cost of the base lining material is different from that of the material used in the side walls, the equation for the required section factor (Eqs. 3 or 4), must be solved such that the cost function (Eq. 9) is minimized. Mathematically, this optimization problem is the same as the classical microeconomic problem of minimizing production costs through input (i.e., capital and labor) substitution (2). Our output is the hydraulic capacity of the channel. Thus, the "production function" is the equation for the section factor. Our "inputs" are the cross-sectional dimensional parameters, B, D, and z.

The basic solution technique involves determining the input mix at which the ratio of the marginal products equals the ratio of the marginal costs. This mix can be determined for any number of inputs utilizing the Lagrange Multiplier technique (2) although the mathematics can become complex.

In order to reduce the complexity of the channel lining problem to one for which a direct algebraic solution can be formulated and solution graphs can be presented, the side slope, z, will be assumed constant. The side slope is chosen to be the constrained parameter since, under most conditions, the choice of side slope is limited to some extent by the soils, building materials, or construction technique. By solving the simplified optimization problem for a number of side slopes, the minimum cost side slope can be iteratively determined.

As was stated, the optimization method involves determining the dimensional combination such that the ratio of the marginal changes in the section factor is equal to the ratio of the marginal changes in the costs, or

$$\frac{\dfrac{\partial(AR^{2/3})}{\partial B}}{\dfrac{\partial(AR^{2/3})}{\partial D}} = \frac{\dfrac{\partial C}{\partial B}}{\dfrac{\partial C}{\partial D}} \quad \text{(10)}$$

and the equation for the section factor (Eq. 3) is satisfied. When Eqs. 3 and 9 are inserted into Eq. 10 and the partial derivatives taken, the relationship can be reduced, through algebraic manipulation, to a quadratic of the form

$$k_1\left(\frac{D}{B}\right)^2 + k_2\left(\frac{D}{B}\right) + k_3 = 0 \quad \text{(11)}$$

in which $$k_1 = 20(z^2+1) - \left[1 + 4\left(\frac{b}{d}\right)\right] 4z(z^2+1)^{1/2} \quad \text{(12)}$$

$$k_2 = \left[1 - \frac{b}{d}\right] 6(z^2+1)^{1/2} - 10z\left(\frac{b}{d}\right) \quad \text{(13)}$$

and $$k_3 = -5\left(\frac{b}{d}\right) \quad \text{(14)}$$

Thus, the B/D ratio required for the solution of Eq. 4 is

$$\frac{B}{D} = \frac{2k_1}{-k_2^2 + \left(k_2^2 + 20\left(\dfrac{b}{d}\right)k_1\right)^{1/2}} \quad \text{(15)}$$

which depends only upon the chosen side slope and the ratio of the unit cost of the base and side material.

By inserting the b/d ratio of the chosen materials and a selected z value into Eqs. 12 and 13, the computed k_1 and k_2 values into Eq. 15, and the derived B/D ratio into Eq. 4, the minimum cost flow depth, D, is determined. This D value times the B/D value will then give the minimum cost bottom width, and the cross-sectional shape is determined. Inserting these dimensional parameters and

the unit costs into Eq. 9 will give the cost per unit length of the lining materials.

Graphical Solution.—A graphical presentation of Eqs. 4 and 15 allows a quick and illustrative determination of the minimum cost dimensions. The curved lines shown in Fig. 2 each represent one value of the section factor which is in turn equal to $Qn/(cS^{1/2})$. The straight lines radiating from the origin are solutions to Eq. 15 for various b/d values. A separate graph is required for each side slope value. Fig. 2 is drawn for a side slope value of 0.5. Any combination of depth and bottom width that falls on a required section factor line fulfills the hydraulic requirements of the design. The B and D values at the point where the section factor line crosses a b/d cost ratio line is the minimum cost design for that material cost ratio.

A straight line drawn tangent to an equi-section factor curve at the point where the proper b/d line crosses the curve passes through parameter combinations with equal total costs. Consequently, the total cost can be calculated by following this tangent from the required section factor curve to one of the axes and calculating the cost based only on the value of the parameter at the intercept. Thus

$$C = bB_o + 2dF(z^2+1)^{1/2} + k \quad \text{(16)}$$

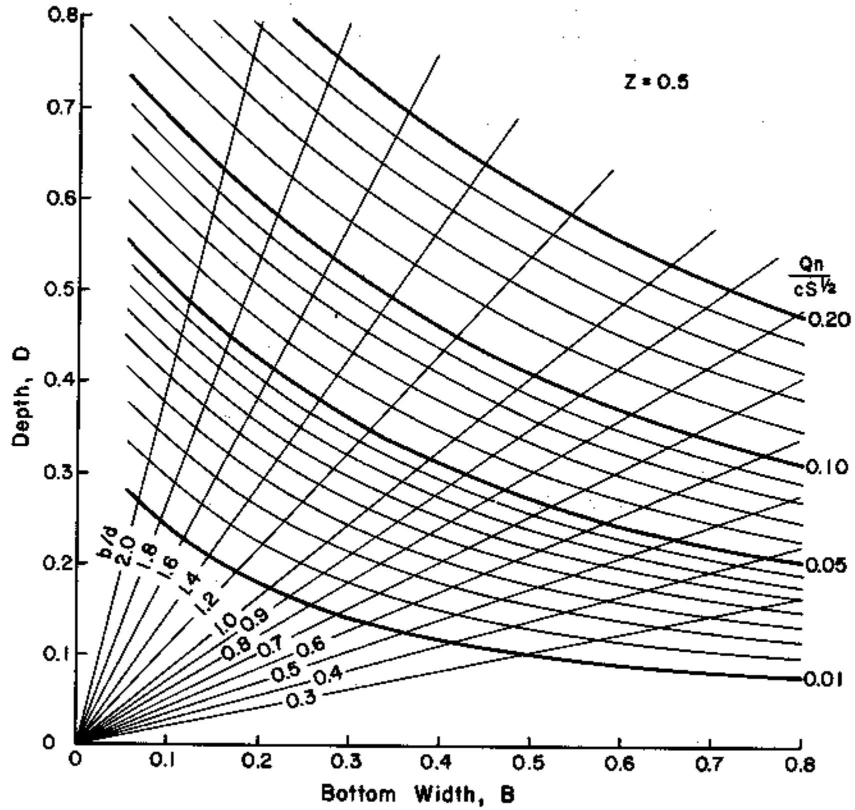

FIG. 2.—Material Cost Minimization Design Solution for Trapezoidal Channel Lining with a Side Slope, z, of 0.50

or $C = 2d(D_o + F)(z^2 + 1)^{1/2} + k$ .... (17)

in which B_o and D_o = the intercept values of the two parameters.

By using this principle, the cost increase of moving along an equi-section factor line away from the optimal design point can be quickly determined from the graph by constructing a line through the chosen design point parallel to the tangent at the optimum point. The difference in B_o values times b or the difference in D_o values times $2d(z^2 + 1)^{1/2}$ is the cost increase. These procedures will be demonstrated with an example.

Sample Application

A lined channel is to be built to carry 2.8 cu ft/sec (0.08 m^3/s) of water on a 0.001 ft/ft (m/m) slope. A roughness coefficient of 0.014 is used, so that the required section factor is 0.83 $ft^{8/3}$ (0.035 $m^{8/3}$). The specified base lining material costs \$0.30 per sq ft (\$3.20 per m^2) at the required thickness, while the material to be used in the side walls costs only \$0.19/sq ft (\$2.00/m^2) at the required thickness, giving a material cost ratio, b/d, of 1.6. Material requirements of the corners will cost \$0.11/ft (\$0.35/m) length while a 6-in. (0.15-m) freeboard is specified. Forms are available with 2:1 side slopes or z = 0.5.

The minimum cost bottom width and depth for the required section factor and given material costs can be determined from Fig. 2 or by using Eqs. 12, 13, 15, and 4. Using the figure, follow the $Qn/(cS^{1/2}) = 0.035\ m^{8/3}$ curve to the left until it crosses the b/d = 1.6 line radiating from the origin. The minimum cost

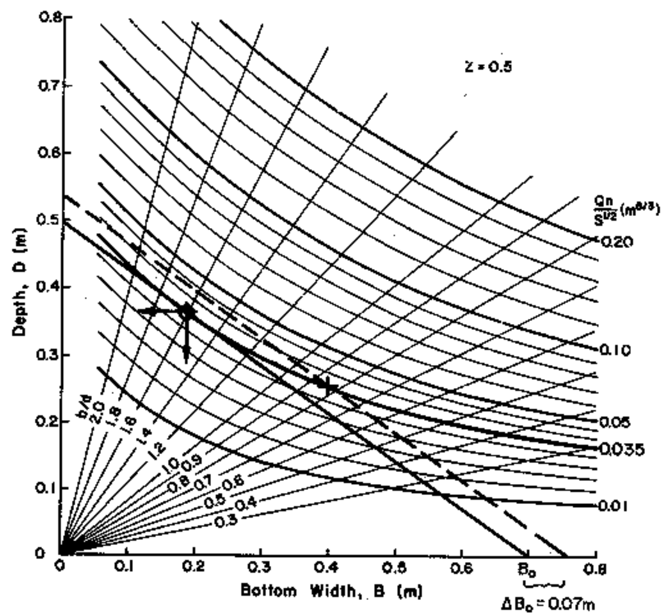

FIG. 3.—Graphical Solution to the Design Example Showing the Minimum Cost Design Point and Equicost Design Lines for the Optimal Design and for B = 0.4 m

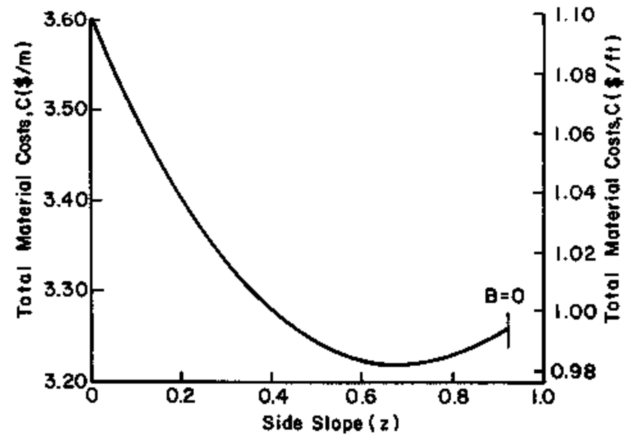

FIG. 4.—Minimum Total Material Cost versus Side Slope for the Design Example

parameters read off the axes are D = 0.36 m (1.18 ft) and B = 0.19 m (0.62 ft). This procedure is shown on Fig. 3. The same values are determined with the equations. These values result in a total cost of (Eq. 9)

$$C = \$3.20 \times 0.19 + \$2.00 \times 2(0.36 + 0.15)(0.5^2 + 1)^{1/2} + \$0.35$$
$$= \$3.24/\text{m}\ (\$1.01/\text{ft}) \quad \text{(18)}$$

The available construction forms have 1.3-ft (0.4-m) bottom widths. The extra material cost of using these forms can be calculated by following the section factor line to B = 0.4 m, reading D = 0.25 m from the ordinate, and inserting these numbers into Eq. 9. However, by constructing a tangent at the optimum design point and a parallel line through this chosen design point, as shown in Fig. 3, the equivalent increase in the base width, ΔB_o, is seen to be 0.07 m, giving a cost increase of $\Delta B_o \times b$ = \$0.22/m (\$0.07/ft) or 7%.

Fig. 4 shows the sensitivity of material costs of the posed example to variations in the side slope. Minimum cost occurs at a side slope value of 0.64, but the cost even at a 3:1 side slope (z = 0.33) is only \$0.09/m (\$0.03/ft) or 3% above this minimum. Note that at side slopes above z = 0.92, the optimum bottom width has decreased to zero and the minimum cost shape is a triangle.

As this example indicates, material costs are not sensitive to cross-sectional dimension variations as long as they do not deviate too far from the optimum dimensions. By use of these design equations or graphs, this extra material cost can easily be calculated so that it can be evaluated relative to other costs of the construction. Also note that the design graphs, which allow a large range of cross-sectional parameter options to be quickly surveyed, can be used to efficiently design any trapezoidal channels regardless of cost considerations.

Summary and Conclusions

A direct algebraic and graphical solution was developed based on both hydraulic and microeconomic theory, which allows the design of lined trapezoidal channels to minimize material costs when bed and side lining costs are different. The method can also be used to evaluate added costs of deviating from optimal designs. Use of the technique shows that moderate deviations from optimal designs do not usually cause a significant increase in costs. Although the benefits gained by minimizing costs may not be large, in the light of the simplicity of the technique, the knowledge gained from the exercise is worthwhile.

Appendix I.—References

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Appendix II.—Notation

The following symbols are used in this paper:

A = cross-sectional flow area;
B = channel bottom width;
B_o = bottom width value at the graph intercept ($D = 0$);
B' = bottom corner width;
b = channel bottom material cost per unit area;
C = channel lining material cost per unit length;
C_b = base material cost per unit length;
C_s = side material cost per unit length;
c = constant in Manning's Equation for conversion to U.S. Customary units (1.486);
D = normal flow depth;
D_o = depth value at the graph intercept ($B = 0$);
d = channel side material cost per unit area;
E = channel wetted side length;
E' = freeboard side length allowance;
F = freeboard allowance;
k = corner material cost per unit length;
k_1, k_2, and k_3 = constants defined by Eqs. 12, 13, and 14;
n = Manning roughness coefficient;
P = wetted perimeter length;
Q = flow rate;
R = hydraulic radius;
S = channel slope;
t_b = thickness of channel bottom lining;
t_s = thickness of channel side lining;
z = channel side slope (horizontal-to-vertical);
μ_b = cost of base lining material per unit volume; and
μ_s = cost of side lining material per unit volume.